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The condensate equation for some Bose systems

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Abstract. The exactness of the Bogoliubov approximation is discussed from the point of view of the condensation properties of the Bose gas. For the imperfect Bose gas, we find that the approximation yields the correct condensate density when the thermodynamic limit is taken by isotropic dilation; this is not the case in general for other ways of going to the infinite volume limit. We also prove the existence of Bose–Einstein condensation for a class of Bose gases with weak interactions and an energy gap.

1. Introduction

Since its introduction in 1947 (Bogoliubov 1947), the Bogoliubov approximation has been used extensively in condensed matter physics, but it was as late as 1968 before Ginibre (1968) provided a mathematical justification of the procedure. At that stage, the way was paved for using Bogoliubov's method to deduce exact results on Bose–Einstein condensation, though very few authors seem to have worked along these lines. In this paper, we shall prove some exact results about the Bogoliubov approximation; our main concern will be the mean-field Bose gas, but we shall also discuss a class of perturbations around this system.

The basic idea of the Bogoliubov approximation is to replace the ground-state annihilation and creation operators a_0, a_0^* in the Hamiltonian H^V of the interacting Bose gas by complex parameters $V^{1/2}\alpha$, $V^{1/2}\bar{\alpha}$, V being the volume of the region containing the system. One obtains in this way a new Hamiltonian $H_B^V(\alpha)$ for each value of α, and a corresponding pressure:

$$P_B^V(\alpha) = (\beta V)^{-1} \log \operatorname{Tr} \exp(-\beta H_B^V(\alpha)). \tag{1}$$

The physical value α_0^V of the parameter α is determined *a posteriori* by the condition that $P_B^V(\alpha)$ attains its maximum at α_0^V:

$$P_B^V(\alpha_0^V) = \sup_\alpha P_B^V(\alpha). \tag{2}$$

We shall refer to (2) as the *condensate equation*, though in the literature (Ginibre 1968) it is the necessary condition

$$(\partial P_B^V(\alpha_0^V)/\partial\alpha) = 0 \tag{3}$$

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which is given this name. Since by gauge symmetry $P_B^V(\alpha)$ is a function of $|\alpha|$ only, we shall take α to be real and positive throughout this paper.

As already mentioned, Ginibre (1968) gave a rigorous proof that for a large class of interactions the approximation is thermodynamically exact, that is to say that $\lim_{V\to\infty} P_B^V(\alpha_0^V)$ yields the exact pressure of the infinite system. A much more involved question is that of the exactness of the Bogoliubov approximation from the point of view of the condensation properties of the system, i.e. the relation of $(\alpha_0^V)^2$ to $\langle a_0^* a_0/V\rangle$.

In fact Ginibre's method can be used to prove that, if one defines

$$\pi^V(\sigma) = (\beta V)^{-1} \log \operatorname{Tr} \exp[-\beta(H^V - \sigma a_0^* a_0)] \tag{4}$$

then $\pi^V(\sigma)$ is asymptotically equal in the thermodynamic limit to the Legendre transform of $-P_B^V(\alpha)$ with respect to α^2, namely:

$$\lim_{V\to\infty}\left\{\pi^V(\sigma) - \sup_\alpha[\sigma\alpha^2 + P_B^V(\alpha)]\right\} = 0. \tag{5}$$

This leads us to expect that the probability distribution of the condensate density $a_0^* a_0/V$ (viewed as a random variable) is asymptotically concentrated on the subdifferential of $\pi(\sigma)$ of $\sigma = 0$ (see Lewis and Pulé (1983a) for a similar discussion on the overall density), and therefore on the set of solutions of the asymptotic condensate equation; in particular, in cases where the solution α_0 of the asymptotic condensate equation is unique, the condensate density should be degenerately distributed at the corresponding value α_0^2. As yet we are not able to prove these conjectures in general, but our claims can be verified rigorously for the imperfect Bose gas, i.e. in the case of a mean-field repulsive interaction (see theorem 1 in § 2 and proposition 3 in § 4).

From this point of view, the next step in the study of Bose–Einstein condensation is to examine the non-zero solutions of equations (2) and (3) and their behaviour as V tends to infinity. A simple gauge argument (Ginibre 1968) shows that $\alpha = 0$ is always a solution of equation (3). It is reasonable to expect that the onset of Bose–Einstein condensation (i.e. the emergence of a non-zero solution) is accompanied by a change in the nature of the stationary point $\alpha = 0$ from a maximum to a minimum of $P_B^V(\alpha)$ (in bifurcation language the trivial solution $\alpha = 0$ becomes unstable when the bifurcation parameter μ crosses a critical value). This, together with bounds which ensure that $P_B^V(\alpha)$ decreases faster than $-\alpha^4$ for α large (Ginibre 1968), suggests the following tentative picture to characterise Bose–Einstein condensation (V large enough, see figure 1). This picture is justified by our study of the finite volume imperfect Bose gas in § 3 (see theorem 2).

But in the infinite volume limit, the Bogoliubov pressure of the imperfect Bose gas becomes concave, with (for μ large enough) a flat part starting at $\alpha = 0$ (see proposition 1). At first sight it might be disappointing that the condensate density cannot be read

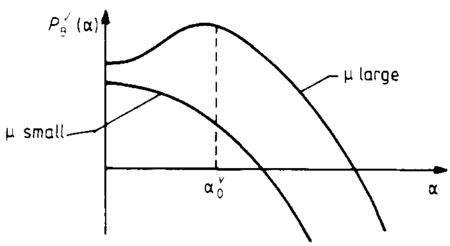

Figure 1. Tentative characterisation of Bose–Einstein condensation.

off the shape of $P_B(\alpha) \equiv \lim_{V\to\infty} P_B^V(\alpha)$. However, this should not come as a surprise in view of the fact that it is possible to exhibit Bose systems which have the same thermodynamic functions, but which differ drastically in their condensation properties (Van den Berg and Lewis 1980). Hence one should not expect that a purely thermodynamic quantity like $P_B(\alpha)$ will prescribe the condensate density; at most it will provide an upper bound on the latter. This also fits in very well with the result of § 2 which shows that the support of the probability distribution of the condensate density is contained in the interval on which $P_B(\alpha)$ is flat (theorem 1). The fact that $\alpha = 0$ is always a global maximum of $P_B(\alpha)$ appears to hold even in interacting Bose gases.

To obtain more detailed information, one has to study equation (2) at finite volume, and different ways of going to the infinite volume limit may then lead to different values of the limiting α_0 on the plateau of P_B (see figure 2).

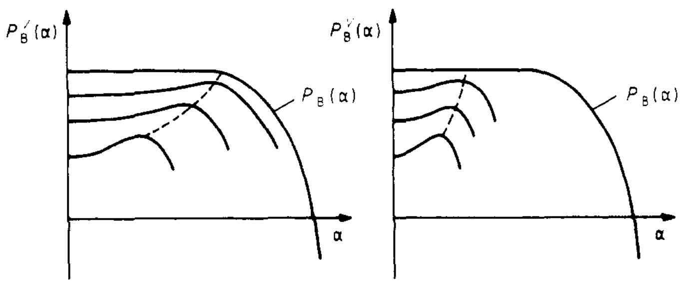

Figure 2. Two ways of approaching the thermodynamic limit leading to different values of α_0.

Now this dependence on the way the thermodynamic limit is taken is a property shared by the exact condensate density. It is thus tempting to conjecture that the limiting value of the solution to the finite volume condensate equation yields the correct condensate density under all circumstances. We show by means of a counter example that it is not so. In detail, we study in § 3 the imperfect Bose gas in a volume V which goes to infinity

(i) in an isotropic way;

(ii) in some prescribed anisotropic way.

In case (i), α_0^V converges to the right-hand endpoint of the plateau of P_B (see theorem 2) and this gives the exact condensate density (which is known from the direct theory (Fannes and Verbeure 1980, Buffet and Pulé 1983)). In case (ii), the limiting value of α_0^V bears no relation to the condensate density (see proposition 2); incidentally, the latter assumes different values in cases (i) and (ii) (see theorem 3).

Hence we have at hand a simple model in which the widely accepted Bogoliubov approximation fails to predict the condensation properties of the Bose gas. The limitations of this method should be kept in mind when applying it to more complicated situations.

We conclude our work with the study of interacting Bose gases with a gap in their single particle energy spectrum. If the interaction is weak enough and the chemical potential large enough, one can show that $P_B^V(\alpha)$ takes its maximum in a region not containing zero, even in the thermodynamic limit (theorem 5). This strongly suggests existence of Bose–Einstein condensation, and we also provide a direct proof of the occurrence of this phenomenon (theorem 4).

Before proceeding to the next section, we make precise the definitions of some quantities discussed above.

Consider a Bose gas contained in Λ^V, a cuboid in $\mathbb{R}^3$ with sides V^{α_1}, bV^{α_2}, cV^{α_3} where

$$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 > 0 \qquad \alpha_1 + \alpha_2 + \alpha_3 = 1 \qquad bc = 1. \tag{6}$$

Let $\mathscr{F}^V = \bigoplus_{n=0}^{\infty} S_n(\otimes^n L^2(\Lambda^V))$ be the symmetric Fock space constructed out of $L^2(\Lambda^V)$. Let N^V be the number operator on $\mathscr{F}^V$, and $a(f)$, $a^*(f)$ the annihilation and creation operators ($f \in L^2(\Lambda^V)$). Denote by ϕ_k^V ($k \geqslant 0$) the eigenfunctions and by E_k^V ($0 < E_0^V < E_1^V \leqslant \ldots$) the eigenvalues of the self-adjoint extension of the operator $-\frac{1}{2}\Delta$ with Dirichlet boundary conditions on $L^2(\Lambda^V)$. We write a_k, a_k^* and N_k^V for $a(\phi_k^V)$, $a^*(\phi_k^V)$ and $a^*(\phi_k^V)a(\phi_k^V)$ respectively.

Now let $\mathscr{M}_0^V$ be the one-dimensional subspace of $L^2(\Lambda^V)$ generated by ϕ_0^V. Then $\mathscr{F}^V = \mathscr{F}_0^V \otimes \mathscr{F}_1^V$, where $\mathscr{F}_0^V$ and $\mathscr{F}_1^V$ are the Fock spaces constructed out of $\mathscr{M}_0^V$ and $(\mathscr{M}_0^V)^\perp$ respectively. For each $\lambda \in \mathbb{C}$, we define a coherent vector C_λ in $\mathscr{F}_0^V$ in the usual way:

$$C_\lambda = \exp(-|\lambda|^2/2) \sum_{l=0}^{\infty} \lambda^l (\phi_0^V)^{\otimes l}/(l!)^{1/2}. \tag{7}$$

Following Ginibre (1968), we associate to each operator A on $\mathscr{F}^V$ its Bogoliubov approximate $A_B(\alpha)$, which is the operator on $\mathscr{F}_1^V$ defined by

$$(\phi, A_B(\alpha)\psi)_{\mathscr{F}_1^V} = (C_{\alpha V^{1/2}} \otimes \phi, A C_{\alpha V^{1/2}} \otimes \psi)_{\mathscr{F}^V} \qquad \forall \phi, \psi \in \mathscr{F}_1^V. \tag{8}$$

Finally, let $H^V(\mu)$ be the grand canonical Hamiltonian of the system (including the term $-\mu N^V$, μ being the chemical potential), and $H_B^V(\mu, \alpha)$ its Bogoliubov approximate. The conditions under which one can make sense of $H_B^V(\mu, \alpha)$ as a self-adjoint operator are discussed in Ginibre (1968); assuming that these conditions are met, it follows that $\exp[-\beta H_B^V(\mu, \alpha)]$ is trace class, and one can define (in complete analogy to the ordinary pressure) the Bogoliubov pressure

$$P_B^V(\mu, \alpha) = (\beta V)^{-1} \log \mathrm{Tr}_{\mathscr{F}_1^V} \exp[-\beta H_B^V(\mu, \alpha)]. \tag{9}$$

Its infinite volume limit (whenever it exists) is denoted by $P_B(\mu, \alpha)$.

2. The probability distribution of the Bose–Einstein condensate in the imperfect gas

Let us briefly recall the main features of the imperfect Bose gas (see Fannes and Verbeure 1980, Buffet and Pulé 1983, Davies 1972). Let $H_0^V = \Sigma_{k \geqslant 0} E_k^V a_k^* a_k$ be the kinetic energy of the particles. Then the Hamiltonian for the imperfect gas is defined as

$$H^V(\mu) = H_0^V + \tfrac{1}{2}a(N^V)^2/V - \mu N^V \qquad a > 0. \tag{10}$$

The presence of the second term embodies the assumption that the interaction energy between two particles is a/V, irrespective of their mutual distance. If $p_0(\mu)$, $\rho_0(\mu)$ are the pressure and density of the infinite free Bose gas at chemical potential $\mu \leqslant 0$ and if ρ_c is the critical density for the same system, the pressure and density for the infinite imperfect Bose gas are given by

$$p(\mu)=\begin{cases}\frac{1}{2}a(\rho_0(\gamma))^2+p_0(\gamma) & \mu\leqslant\mu_c\\ \mu^2/2a+p_0(0) & \mu>\mu_c\end{cases}\tag{11}$$

$$\rho(\mu)=\begin{cases}\rho_0(\gamma) & \mu\leqslant\mu_c\\ \mu/a & \mu>\mu_c\end{cases}\tag{12}$$

where $\mu_c=a\rho_c$ and γ is the unique (μ dependent) solution of the equation $\gamma=\mu-a\rho_0(\gamma)$.

Suppose that we remove the lowest energy level E_0^V from the system described above; we would then obtain another system which has the same pressure and density in the thermodynamic limit as the original case. We shall place a tilde over all quantities pertaining to this modified system, in particular:

$$\tilde{N}^V=N^V-N_0^V \qquad \tilde{H}^V(\mu)=H_0^V-E_0^VN_0^V+(a/2V)(\tilde{N}^V)^2-\mu\tilde{N}^V.\tag{13}$$

The Bogoliubov approximation to the Hamiltonian is then

$$H_B^V(\mu,\alpha)=(E_0^V-\mu)\alpha^2V+\tfrac{1}{2}a\alpha^4V+\tfrac{1}{2}a\alpha^2+\tilde{H}^V(\mu-a\alpha^2)\tag{14}$$

and the Bogoliubov pressure is therefore (with $\tilde{p}^V(\mu)$ denoting the pressure of the modified system):

$$P_B^V(\mu,\alpha)=-(E_0^V-\mu)\alpha^2-\tfrac{1}{2}a\alpha^4-a\alpha^2/2V+\tilde{p}^V(\mu-a\alpha^2)\tag{15}$$

from which it follows immediately that

$$P_B(\mu,\alpha)=\mu\alpha^2-\tfrac{1}{2}a\alpha^4+p(\mu-a\alpha^2)\tag{16}$$

with p as in (11).

$P_B(\mu,\alpha)$ has the following properties.

Proposition 1.

(i) $\alpha\to P_B(\mu,\alpha)$ is concave and non-increasing;

(ii) for $\mu<\mu_c:\alpha\to P_B(\mu,\alpha)$ is strictly concave and decreasing;

(iii) for $\mu\geqslant\mu_c:\alpha\to P_B(\mu,\alpha)$ is constant (and equal to $p(\mu)$) for $\alpha\leqslant(\rho(\mu)-\rho_c)^{1/2}$, $\alpha\to P_B(\mu,\alpha)$ is strictly concave and decreasing for $\alpha>(\rho(\mu)-\rho_c)^{1/2}$.

Proof. The proof follows immediately from the following properties of $\rho(\mu)$:

$$\begin{aligned}&\rho(\mu)>\mu/a && \text{for } \mu<\mu_c\\ &\rho(\mu)=\mu/a && \text{for } \mu\geqslant\mu_c\end{aligned}\tag{17}$$

$$\mathrm{d}\rho(\mu)/\mathrm{d}\mu<1/a \qquad \text{for } \mu<\mu_c.\tag{18}$$

Now, denote by $Q^V(n)$ the orthogonal projection onto the subspace of $\mathscr{F}^V$ with exactly n particles in the state ϕ_0^V. We can consider N_0^V/V as a random variable and define its corresponding probability measure $\mathbb{P}_0^V$ by

$$\mathbb{P}_0^V\{x\}=\text{Probability}\{N_0^V/V=x\}=\langle Q^V(Vx)\rangle=\frac{\mathrm{Tr}_{\mathscr{F}^V}\exp(-\beta H^V(\mu))Q^V(Vx)}{\mathrm{Tr}_{\mathscr{F}^V}\exp(-\beta H^V(\mu))}.\tag{19}$$

In the following theorem we prove that in the thermodynamic limit the probability measure $\mathbb{P}_0^V$ is concentrated on the set of values α which maximise $P_B(\mu,\alpha)$. The technique we use is an adaptation of the one used in Lewis and Pulé (1983a) for the probability measure corresponding to the particle density.

Theorem 1. If α_1 is such that $P_B(\mu, \alpha_1) < \sup_\alpha P_B(\mu, \alpha) = P_B(\mu, 0) = P(u)$ then $\lim_{V\to\infty} \mathbb{P}_0^V\{N_0^V / V \geq \alpha_1\} = 0$.

Proof. Let

$$\pi^V(\sigma) = (\beta V)^{-1} \log \operatorname{Tr}_{\mathscr{F}^V} \exp(-\beta(H^V(\mu) - \sigma N_0^V)]. \tag{20}$$

Then

$$\pi^V(\sigma) = \beta V^{-1} \log \sum_{n=0}^{\infty} \exp(\beta\sigma n) \operatorname{Tr}_{\mathscr{F}^V} Q^V(n) \exp[-\beta H^V(\mu)]. \tag{21}$$

Since N_0^V commutes with $H^V(\mu)$

$$\operatorname{Tr}_{\mathscr{F}^V} Q^V(n) \exp[-\beta H^V(\mu)] = \operatorname{Tr}_{\mathscr{F}^V} \exp[-\beta Q^V(n) H^V(\mu)]. \tag{22}$$

It is easy to verify that

$$Q^V(n) H^V(\mu) = Q^V(n)|_{\mathscr{F}_0^V} \otimes \{H_B^V(\mu, (n/V)^{1/2}) - \tfrac{1}{2}a(n/V)\Pi_{\mathscr{F}_1^V}\}. \tag{23}$$

Hence

$$\operatorname{Tr}_{\mathscr{F}^V} Q(n) \exp(-\beta H^V(\mu)) = \exp\{\beta V[P_B^V(\mu, (n/V)^{1/2}) + an/2V^2]\} \tag{24}$$

and

$$\pi_V(\sigma) = \frac{1}{\beta V} \log \sum_{n=0}^{\infty} \exp(\beta V)\left[\frac{\sigma n}{V} + \frac{a}{2V}\frac{n}{V} + P_B^V\left(\mu, \left(\frac{n}{V}\right)^{1/2}\right)\right] \tag{25}$$

$$= \frac{1}{\beta V} \log \int_{[0,\infty)} \exp\left[\beta V\left(\sigma x + \frac{a}{2V}x + P_B^V(\mu, x^{1/2})\right)\right] \mathrm{d}m(Vx) \tag{26}$$

where m is the counting measure.

Using the version of the Laplace theorem stated in the appendix, we have immediately that

$$\pi(\sigma) \equiv \lim_{V\to\infty} \pi^V(\sigma) = \sup_{x\geq 0}(\sigma x + P_B(\mu, x^{1/2})) = \sup_\alpha(\sigma\alpha^2 + P_B(\mu, \alpha)) \tag{27}$$

and therefore

$$\pi'_+(0) = \sup\{\alpha' : P_B(\mu, \alpha') = \sup_\alpha P_B(\mu, \alpha)\} = (\max\{0, \rho(\mu) - \rho_c\})^{1/2} \tag{28}$$

where $\pi'_+(0)$ denotes the right-hand derivative of π at 0. Moreover for $\sigma > 0$, $\pi(\sigma)$ is continuously differentiable and strictly convex. Therefore, we can find $\delta > 0$ such that $\pi'(\delta) < \alpha_1$ and hence $\pi(\delta) - \pi(0) < \alpha_1\delta$. Now

$$\begin{aligned}\mathbb{P}_0^V[N_0^V / V \geq \alpha_1] &= \int_{[\alpha_1,\infty)} \mathrm{d}\mathbb{P}_0^V(\alpha) \\ &\leq \int_{[\alpha_1,\infty)} \exp[-\beta(\alpha_1 - \alpha)\delta V]\, \mathrm{d}\mathbb{P}_0^V(\alpha) \leq \int_{[0,\infty)} \exp[-\beta(\alpha_1 - \alpha)\delta V]\, \mathrm{d}\mathbb{P}_0^V(\alpha) \\ &= \exp[-\beta V(\delta\alpha_1 - \pi^V(\delta) + \pi^V(0))]\end{aligned}$$

which tends to 0 as V tends to infinity.

Before proceeding to the next section, we remark that, as for the particle density (see Lewis and Pulé 1983a), $\mathbb{P}_0^V[N_0^V / V \geq \alpha_1]$ decreases exponentially with the volume.

3. The condensate equation for finite volume

Since $P_B^V(\alpha, \mu)$ tends to $-\infty$ as $\alpha \to \infty$, it must attain its supremum at least at one point and at this point equation (3) must be satisfied. For the imperfect gas (3) becomes (with $\tilde{\rho}^V_{(\mu)} = d\tilde{p}^V(\mu)/d\mu$)

$$2\alpha[(\mu - a\alpha^2) - a\tilde{\rho}^V(\mu - a\alpha^2) - E_0^V - a/2V] = 0. \tag{29}$$

We shall study equations (2) and (29) in two cases differing in the way that the region Λ^V becomes infinite. In the first case (isotropic dilation) we take $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ and b, c distinct and not equal to 1 (see (6)), so that the eigenvalues $\{E_k^V\}$ are related to $\{E_k^1\}$ by $E_k^V = V^{-2/3}E_k^1$ and $E_1^1 < E_2^1$. The case where $E_1^1 = E_2^1$ requires a separate but similar analysis and yields the same conclusions. For this family of regions, the imperfect gas has the property that (Fannes and Verbeure 1980, Buffet and Pulé 1983)

$$\lim_{V\to\infty} \left\langle \frac{N_0^V}{V} \right\rangle_\mu^{gc} = \begin{cases} 0 & \text{for} \quad \mu \leq \mu_c \\ \rho(\mu) - \rho_c & \text{for} \quad \mu > \mu_c \end{cases}, \tag{30}$$

In the following theorem we prove that any sequence of solutions of (2) also gives this value in the limit.

Theorem 2. Consider the mean-field Bose gas in an isotropically dilated box. Let $\alpha_0^V(\mu)$ be a point at which $p_B^V(\alpha, \mu)$ attains its supremum. Then

$$\lim_{V\to\infty} \alpha_0^V(\mu) = \begin{cases} 0 & \mu \leq \mu_c & (31a) \\ (\rho(\mu) - \rho_c)^{1/2} & \mu > \mu_c & (31b) \end{cases}.$$

Proof. Put

$$g^V(\sigma) = \sigma - a\tilde{\rho}^V(\sigma) - E_0^V - a/2V. \tag{32}$$

Then, with (15)

$$\partial P_B^V(\mu, \alpha)/\partial\alpha = 2\alpha g^V(\mu - a\alpha^2).$$

Now from (17) and the uniformity (on compact intervals) of the convergence of $\tilde{\rho}^V(\sigma)$ to $\rho(\sigma)$ we deduce that for any $\varepsilon > 0$

$$g^V(\sigma) < 0 \qquad \text{when } \sigma < \mu_c - a\varepsilon \text{ and } V \text{ is large enough.}$$

Hence, for $\alpha^2 > (\mu - \mu_c)/a + \varepsilon$ and V large enough:

$$\partial P_B^V(\mu, \alpha)/\partial\alpha < 0. \tag{33}$$

This yields two conclusions:

(i) if $\mu < \mu_c$, $\lim_{V\to\infty} \alpha_0^V = 0$, proving (31$a$);

(ii) if $\mu \geq \mu_c$, $(\alpha_0^V)^2 < \rho(\mu) - \rho_c + \varepsilon$ for V large enough.

To have (31b), it is sufficient to show that the following inequality holds when $\mu > \mu_c$ and V is large enough:

$$(\alpha_0^V)^2 > \rho(\mu) - \rho_c - \varepsilon. \tag{34}$$

This will hold if we can show that for V large enough $g^V(\sigma)$ is positive for all values of σ in $[\mu_c+a\varepsilon, \mu]$. This in turn will follow if we can prove that $|V(\sigma-a\tilde{\rho}^V(\sigma)-E_1^V)|$ is uniformly bounded for all

$$\sigma\in[\mu_c+a\varepsilon, \mu] \tag{35}$$

because of the identity:

$$g^V(\sigma)=V^{-2/3}[(E_1^1-E_0^1)+V^{2/3}(\sigma-a\tilde{\rho}^V(\sigma)-E_1^V)-\tfrac{1}{2}aV^{-1/3}]. \tag{36}$$

We now prove (35): with $\sigma>\mu_c$ one has $\rho(\sigma)=\sigma/a$, and thus

$$V\left(\tilde{\rho}_V(\sigma)-\frac{\sigma}{a}+\frac{E_1^V}{a}\right)=\frac{V\sum_{n=0}^{\infty}(n/V-\rho(\sigma)+E_1^V/a)\tilde{Z}_n^V\exp[-\frac{1}{2}\beta a_V(n/V-\rho(\sigma))^2]}{\sum_{n=0}^{\infty}\tilde{Z}_n^V\exp[-\frac{1}{2}\beta aV(n/V-\rho(\sigma))^2]} \tag{37}$$

where $\tilde{Z}_n^V$ denotes the canonical partition function of the related free system. If

$$P^V=-(\beta V)^{-1}\sum_{k\geq 2}\log\{1-\exp[-\beta(E_k^V-E_1^V)]\tag{38}$$

we know from (Buffet and Pulé 1982, theorem 4) that

$$\frac{1}{V^{1/2}}\exp[-\beta V(P^V-\rho(\sigma)E_1^V)]\sum_{n=0}^{\infty}\tilde{Z}_n^V\exp\left[-\tfrac{1}{2}a\beta V\left(\frac{n}{V}-\rho(\sigma)\right)^2\right] \tag{39}$$

converges to $\int_{-\infty}^{+\infty}dx\exp(-\frac{1}{2}a\beta x^2)>0$ uniformly for $\sigma\in[\mu_c+\varepsilon, \mu]$.

It remains to prove that

$$A^V\equiv\sum_{n=0}^{\infty}V^{1/2}\left(\frac{n}{V}-\rho(\sigma)+\frac{E_1^V}{a}\right)\tilde{Z}_n^V\exp\left[-\tfrac{1}{2}a\beta V\left(\frac{n}{V}-\rho(\sigma)\right)^2\right]$$
$$\times\exp[-\beta V(P^V-\rho(\sigma)E_1^V)] \tag{40}$$

is uniformly bounded in modulus for $\sigma\in[\mu_c+a\varepsilon, \mu]$ and for V sufficiently large. Take $n_0\in\mathbb{N}$ such that $(n_0-1)/V-\rho(\sigma)+E_1^V/a<0\leq n_0/V-\rho(\sigma)+E_1^V/a$. Let δ^V be equal to $V^{1/2}(n_0/V-\rho(\sigma)+E_1^V/a)$ and let M_V be the measure concentrated on the points $\{V^{1/2}(n/V-\rho(\sigma)+E_1^V/a)-\delta^V: n\in\mathbb{N}\}$ which gives weight $1/V^{1/2}$ to each of these points. Then, since δ^V is chosen such that for any $n\in\mathbb{N}$, $V^{1/2}(n/V-\rho^V(\sigma)+E_1^V/a)-\delta^V$ is of the form $m/V^{1/2}$ with $m\in\mathbb{Z}$, A^V can be written as

$$A^V=V^{1/2}\exp\left(\frac{\beta V(E_1^V)^2}{2a}\right)\int_{[-V^{1/2}(\rho(\sigma)-E_1^V/a)-\delta^V, \infty)}(x+\delta^V)$$
$$\times\exp\left\{-\beta V\left[f^V\left(\rho(\sigma)-\frac{E_1^V}{a}+\frac{x+\delta^V}{V^{1/2}}\right)\right]\right\}\exp[-\tfrac{1}{2}a\beta(x+\delta^V)^2]M_V(dx) \tag{41}$$

where $f^V(y)=\tilde{q}^V(y)-E_1^Vy+P^V$, $\tilde{q}^V$ being the canonical free energy obtained from $\tilde{Z}_n^V$, i.e. $\tilde{Z}_n^V=\exp[-\beta V\tilde{q}^V(n/V)]$. If $b=\frac{1}{2}(\rho(\sigma)-\rho_c)$, then the contributions to A^V coming from the integration over $[-V^{1/2}(\rho(\sigma)-E_1^V/a)-\delta^V, -V^{1/2}b)$ and $(V^{1/2}b, \infty)$ tend to zero since $f^V(y)\geq 0$ (see Buffet and Pulé 1983, theorem 4). The remainder can be written as

$$V^{1/2}\exp\left(\frac{\beta V(E_1^V)^2}{2a}-\tfrac{1}{2}\beta a(\delta^V)^2\right)$$
$$\times\int_{(0,V^{1/2}b]}\left[\!\left[(x+\delta^V)\exp\left\{-\beta V\left[f^V\left(\rho(\sigma)-\frac{E_1^V}{a}+\frac{x}{V^{1/2}}+\frac{\delta^V}{V^{1/2}}\right)+\frac{ax\delta}{V}\right]\right\}\right.\right.$$
$$-(x-\delta^V)\exp\{-\beta V[f^V(\rho(\sigma)-E_1^V/a-x/V^{1/2}+\delta^V/V^{1/2})$$
$$\left.\left.-ax\delta^V/V]\}\right]\!\right]\exp\left(-\frac{\beta a x^2}{2}\right)M_V\,(\mathrm{d}x)$$
$$+\exp\left(\frac{\beta V(E_1^V)^2}{2a}-\frac{\beta a(\delta^V)^2}{2}\right)\delta^V$$
$$\times\exp\left\{-\beta V\left[f^V\left(\rho(\sigma)-\frac{E_1^V}{a}+\frac{\delta^V}{V^{1/2}}\right)\right]\right\}. \qquad (42)$$

Now, using the mean value theorem, the convexity (Davies 1972) and the positivity of f^V (Buffet and Pulé 1982), and the fact that $\delta^V<1/V^{1/2}$, we find the following upper bound for the absolute value of (42):

$$\exp\frac{\beta V(E_1^V)^2}{2a}\left[-V\beta f_+^{V'}\left(\rho_c+\frac{\varepsilon}{2}-\frac{E_1^V}{a}\right)+\frac{\beta a}{V^{1/2}}\right]\int_0^\infty 2x^2\exp(-\tfrac{1}{2}\beta a x^2)$$
$$\times\exp(\beta a x/V^{1/2})M_V(\mathrm{d}x)$$
$$+2\exp(\beta V(E_1^V)^2/2a)\int_0^\infty\exp(\beta a x/V^{1/2})\exp(-\tfrac{1}{2}\beta a x^2)M_V(\mathrm{d}x)$$
$$+V^{-1/2}\exp[\beta V(E_1^V)^2/2a] \qquad (43)$$

where $f_+^{V'}$ denotes the right-hand derivative of f^V.

But $Vf^V(y)$ is a convex function and for $y>\rho_c$, since $Vf^V(y)\to 0$ (Buffet and Pulé (1983) formulae (25), (34)), we have $Vf_+^{V'}(y)\to 0$ (Griffiths 1964, Griffiths' lemma). Therefore (43) converges, but, since (43) is also independent of σ, we have proved that A^V is uniformly bounded in modulus for V large enough and thus also $V(\sigma-a\tilde{\rho}^V(\sigma)-E_1^V)$.

Next we turn to an anisotropic way of going to the infinite volume limit; specifically we choose the parameters in (6) to be

$$\alpha_1=\tfrac{1}{2}\qquad \alpha_3=\tfrac{1}{2}-\alpha_2\qquad \tfrac{1}{4}\leqslant\alpha_2<\tfrac{1}{2}. \qquad (44)$$

The grand canonical free Bose gas in regions of this type has been studied in great detail by Van den Berg and Lewis (1980). In particular, it is known that for mean density $\bar{\rho}>\rho_c$, the limiting value of $\langle N_0^V/V\rangle_{\bar{\rho}}^{\text{gc,free}}$ is the solution ρ_0 of the equation

$$\sum_{k=0}^{\infty}\frac{\rho_0}{\rho_0(\varepsilon_k-\varepsilon_0)+1}=\bar{\rho}-\rho_c \qquad (45)$$

where the numbers $\varepsilon_k=\tfrac{1}{2}(k+1)^2\pi^2$ are the eigenvalues of $-\tfrac{1}{2}\mathrm{d}^2/\mathrm{d}x^2$ on $[0,1]$ with Dirichlet boundary conditions. Our next theorem solves the same problem for the imperfect Bose gas. Now, because of the strict equivalence of ensembles for the imperfect Bose gas (Buffet and Pulé 1983) and of the peculiar nature of the mean-field interaction, this problem amounts to solving the canonical free Bose gas at the same density (see Buffet and Pulé 1983). Solving this problem, we find that the answer differs from (45); this is yet another example of a pathology associated with the grand canonical free Bose gas (see Buffet and Pulé 1983, Ziff *et al* 1977).

Theorem 3. For the imperfect Bose gas in the family of cuboids described above (see (44)) and $\mu > \mu_c$:

$$\lim_{V\to\infty}\left\langle\frac{N_0^V}{V}\right\rangle_{\rho V}^{\mathrm{gc,imp}}=\frac{\sum_{k=0}^{\infty} b_k\{\eta_k(\rho(\mu)-\rho_c)-1+\exp[-\eta_k(\rho(\mu)-\rho_c)]\}}{\sum_{k=0}^{\infty} b_k\eta_k\{1-\exp[-\eta_k(\rho(\mu)-\rho_c)]\}} \tag{46}$$

where

$$\eta_k=\beta(\varepsilon_k-\varepsilon_0) \qquad b_k=\eta_k^{-1}\prod_{j\neq k}(1-\eta_k/\eta_j)^{-1} \tag{47}$$

and $\varepsilon_k=\frac{1}{2}(k+1)^2\pi^2$ as above.

Proof. As mentioned above, it is enough to compute the limit of the average of N_0^V/V in the canonical free gas. Suppose $\rho>\rho_c$ and define the distribution function $K^V(x)$ as in Buffet and Pulé (1983)

$$K^V(x)=\begin{cases} Z_r^V \exp[-\beta(V\tilde{P}^V-rE_0^V)] & r/V<x\leq(r+1)/V \qquad (r\in\mathbb{N}) \\ 0 & x\leq 0\end{cases} \tag{48}$$

where

$$\tilde{P}^V=-(\beta V)^{-1}\sum_{k=1}^{\infty}\log\{1-\exp[-\beta(E_k^V-E_0^V)]\}. \tag{49}$$

Then

$$\int_{[0,\infty)}\exp(-\lambda x)K^V(\mathrm{d}x)=\exp\left[-\sum_{k=1}^{\infty}\log\left(1+\frac{1-\exp(-\lambda/V)}{\exp[\beta(E_k^V-E_0^V)]-1}\right)\right]. \tag{50}$$

Put

$$F_V(\eta)=(1/V)\max\{k:\eta_k^V\leq\eta\} \tag{51}$$

where $\eta_k^V=E_k^V-E_0^V$. Then

$$\sum_{k=1}^{\infty}\log\left(1+\frac{1-\exp(-\lambda/V)}{\exp[\beta(E_k^V-E_0^V)]-1}\right)$$

can be written as

$$V\int_{(0,\infty)}F_V(\mathrm{d}\eta)\log\left(1+\frac{1-\exp(-\lambda/V)}{\exp(\beta\eta)-1}\right). \tag{52}$$

We split this integral up in an integral over the interval $(V^{2\alpha_2-1},\infty)$ and one over the interval $(0, V^{2\alpha_2-1}]$. First, consider the integral over the interval $(V^{2\alpha_2-1},\infty)$. It can be shown: (i) that for V large enough and all $\eta>V^{2\alpha_2-1}$ there exists a constant c such that $F_V(\eta)\leq c\eta^{3/2}$ and (ii) $F(\eta)=\lim_{V\to\infty}F_V(\eta)=(\sqrt{2}/3\pi^2)\eta^{3/2}$. Moreover, for all positive x we have $x-\frac{1}{2}x^2\leq\log(1+x)<x$.

Using these results, we then find

$$\lim_{V\to\infty} V\int_{(V^{2\alpha_2-1},\infty)} F_V(\mathrm{d}\eta)\log\left(1+\frac{1-\exp(-\lambda/V)}{\exp(\beta\eta)-1}\right)=\lambda\rho_c \tag{53}$$

where ρ_c is the critical density of the ideal Bose gas and is also equal to

$$\int_0^\infty F(\mathrm{d}\eta)\frac{1}{\exp(\beta\eta)-1}.$$

Now, consider the integral over the interval $(0, V^{2\alpha_2-1}]$. It is easy to see $\lim_{V\to\infty} VF_V(\mathrm{d}\eta/V)$ exists and is equal to the measure concentrated on the points η_k/β and which gives weight 1 to each of these points. Then

$$\begin{aligned}\lim_{V\to\infty} V\int_{(0,V^{2\alpha_2-1}]} &F_V(\mathrm{d}\eta)\log\left(1+\frac{1-\exp(-\lambda/V)}{\exp(\beta\eta)-1}\right)\\ &=\lim_{V\to\infty}\int_{(0,V^{2\alpha_2-1}]} VF_V\left(\frac{\mathrm{d}\eta}{V}\right)\log\left(1+\frac{1-\exp(-\lambda/V)}{\exp[\beta(\eta/V)]-1}\right)\\ &=\sum_{k=1}^{\infty}\log\left(1+\frac{\lambda}{\eta_k}\right).\end{aligned}$$

More details can be found in Lewis *et al* (1983). Therefore

$$\begin{aligned}\lim_{V\to\infty}\int_{[0,\infty)}\exp(-\lambda x)K^V(\mathrm{d}x)&=\exp\left\{-\left[\lambda\rho_c+\sum_{k=1}^{\infty}\log\left(1+\frac{\lambda}{\eta_k}\right)\right]\right\}\\ &=\int_0^\infty \exp(-\lambda x)K(\mathrm{d}x)\end{aligned} \tag{54}$$

where

$$K(x)=\begin{cases}0 & x\leqslant\rho_c\\ \sum_{k=1}^{\infty} b_k\eta_k\{1-\exp[-\eta_k(x-\rho_c)]\} & x>\rho_c.\end{cases} \tag{55}$$

That $K(x)$ is indeed equal to (55) can be verified by induction on the number of levels η_k and using the fact that the Laplace transform of a convolution product is the product of the Laplace transforms, i.e.

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}x_1\ldots\int_{-\infty}^{+\infty}\mathrm{d}x_n\,&\exp(-\lambda x_1)f_1(x_1-x_2)\ldots f_{n-1}(x_{n-1}-x_n)f_n(x_n)\\ &=\prod_{i=1}^{n}\left(\int \mathrm{d}x\,\exp(-\lambda x)f_i(x)\right).\end{aligned}$$

Now since $K(x)$ is continuous, we have $\lim_{V\to\infty}K_V(x)=K(x)$ for all $x\in\mathbb{R}$, and the measure $K_V(\mathrm{d}x)$ also converges to $K(\mathrm{d}x)$ for any bounded interval (Feller 1966). As in Buffet and Pulé (1983), we have for the free canonical Bose at density ρ:

$$\left\langle\frac{N_0^V}{V}\right\rangle_{\rho V}^{\mathrm{can}}=\frac{\int_{[0,\rho+1/V]}(\rho-x)K^V(\mathrm{d}x)}{K^V(\rho+1/V)} \tag{56}$$

which, for $\rho > \rho_c$, tends to

$$\frac{1}{K(\rho)} \int_0^\rho (\rho - x) K(\mathrm{d}x). \tag{57}$$

This can be evaluated exactly; we find

$$\lim_{V\to\infty} \left\langle \frac{N_0^V}{V} \right\rangle_{\rho V}^{\text{gc,imp}} = \lim_{V\to\infty} \left\langle \frac{N_0^V}{V} \right\rangle_{\rho V}^{\text{can,free}} = \frac{1}{K(\rho)} \sum_{k=1}^{\infty} b_k\{\eta_k(\rho-\rho_c) - 1 + \exp[-\eta_k(\rho-\rho_c)]\}. \tag{58}$$

We now turn to the corresponding problem in the Bogoliubov approximation. Most of the proof of theorem 2 remains valid for our new family of cuboids (44). In particular, (36) becomes

$$g^V(\sigma) = (1/V)[(\varepsilon_1 - \varepsilon_0) + V(\sigma - a\tilde{\rho}^V(\sigma) - \varepsilon_1/V) - \tfrac{1}{2}a]. \tag{59}$$

The expression (39) still converges uniformly (for σ in compacts sets) to a non-zero limit, namely

$$K(\rho(\sigma)) \int_{-\infty}^{\infty} \mathrm{d}x \exp(-\tfrac{1}{2}a\beta x)^2 \geqslant K(\rho_c + \varepsilon) \int_{-\infty}^{\infty} \mathrm{d}x \exp(-\tfrac{1}{2}a\beta x^2) \tag{60}$$

(assuming $\rho(\sigma) \geqslant \rho_c + \varepsilon$).

The contributions to (41) which come from the integration over $[-V^{1/2}(\rho(\sigma) - E_1^V/a) - \delta^V, -V^{1/2}b]$ and $[V^{1/2}b, \infty)$ still tend to zero uniformly (for σ in compact sets). The remainder is:

$$\begin{aligned}
&\exp[\beta(\varepsilon_1)^2/2aV]V^{1/2}\exp[-\tfrac{1}{2}a\beta(\delta^V)^2] \\
&\quad\times \int_{(0,bV^{1/2})} \delta^V \left[\!\!\left[\exp\left\{-\beta V\left[f^V\left(\rho(\sigma) - \frac{\varepsilon_1}{aV} + \frac{x}{V^{1/2}} + \frac{\delta^V}{V^{1/2}}\right) + \frac{ax\delta^V}{V}\right]\right\}\right.\right. \\
&\quad\left.\left. + \exp\left\{-\beta V\left[f^V\left(\rho(\sigma) - \frac{\varepsilon_1}{aV} - \frac{x}{V^{1/2}} + \frac{\delta^V}{V^{1/2}}\right) - \frac{ax\delta^V}{V}\right]\right\}\right]\!\!\right] \\
&\quad\times \exp(-\tfrac{1}{2}\beta a x^2)\, M_V(\mathrm{d}x) + \exp[\beta(\varepsilon_1)^2/2aV]\delta^V \\
&\quad\times \exp -[a\beta(\delta^V)^2/2] \exp\{-\beta V[f^V(\rho(\sigma) - \varepsilon_1/aV + \delta^V)]\} \\
&\quad + \exp[\beta(\varepsilon_1)^2/2aV]V^{1/2}\exp[-\tfrac{1}{2}a\beta(\delta^V)^2] \\
&\quad\times \int_{(0,bV^{1/2})} x \left[\!\!\left[\exp\left\{-\beta V\left[f^V\left(\rho(\sigma) - \frac{\varepsilon_1}{aV} + \frac{x}{V^{1/2}} + \frac{\delta^V}{V^{1/2}}\right) + \frac{ax\delta^V}{V}\right]\right\}\right.\right. \\
&\quad\left.\left. - \exp\left\{-\beta V\left[f^V\left(\rho(\sigma) - \frac{\varepsilon_1}{aV} - \frac{x}{V^{1/2}} + \frac{\delta^V}{V^{1/2}}\right) + \frac{ax\delta^V}{V}\right]\right\}\right]\!\!\right] \\
&\quad\times \exp\left(-\frac{a\beta x^2}{2}\right) M_V(\mathrm{d}x) \\
&\quad\equiv B_V + C_V + D_V.
\end{aligned}$$

One can then prove the following results. For any interval of $[\rho_c+\varepsilon,\infty)$ and for any δ, there exists a V_0 such that we have for all $V>V_0$ and all σ in this interval,

$$|B_V - V^{1/2}\delta^V(2\pi/a\beta)^{1/2}K(\rho(\sigma))|<\tfrac{1}{3}\delta \qquad (61)$$

$$|C_V|<\tfrac{1}{3}\delta \qquad (62)$$

$$|D_V-[2\pi/(a\beta)^3]^{1/2}K'(\rho(\sigma))-V^{1/2}\delta^V(2\pi/a\beta)^{1/2}K(\rho(\sigma))|<\tfrac{1}{3}\delta \qquad (63)$$

where (63) is derived using the mean value theorem.

Using (61), (62) and (63), we thus get that, given $a>0$, $\delta>0$ and an interval $[c,d]\subseteq[\rho_c+\varepsilon,\infty]$, we can find a V_0 such that for all $V>V_0$

$$V(\sigma-a\tilde{\rho}^V(\sigma)+E_1^V)\leqslant a\delta/[K(\rho_c+\varepsilon)(2\pi/a\beta)^{1/2}-\delta] \qquad (64)$$

for all $\sigma\in[c,d]$. Here we also used that $K'(x)\geqslant 0\ \forall x$.

Now by choosing $\delta=a^{-2}$ and a large, we can make the upper bound (64) arbitrarily close to 0 and consequently we can make

$$(\varepsilon_1-\varepsilon_0)+V(\sigma-a\tilde{\rho}^V(\sigma)+E_1^V)-\tfrac{1}{2}a$$

negative. The next proposition then follows immediately.

Proposition 2. For the imperfect Bose gas in the family of cuboids described above (see (44)) and interaction parameter a large enough, the only possible limit points of $\{\alpha_0^V\}$ are 0 and $(\rho(\mu)-\rho_c)^{1/2}$; consequently the limiting value of $\langle N_0^V\rangle/V$ is not a limit point of $\{(\alpha_0^V)^2\}$.

4. A weakly interacting Bose gas with an energy gap

The introduction of a gap in the one-particle energy spectrum of the Bose gas can be justified in several ways. First one can note that such a gap can be produced by the use of attractive boundary conditions in the definition of the one-particle free Hamiltonian (Robinson 1976, Landau and Wilde 1979, Van den Berg 1982). Next, it might turn out that the interparticle interaction will cause a gap to appear in the excitation spectrum; as we admit *a priori* its presence we only deal with the part of the problem which consists in analysing the consequences of the existence of a gap.

The free system with a gap is conveniently represented by the Hamiltonian

$$H_{0,\Delta}^V=H_0^V-\Delta N_0^V \qquad \Delta>0 \qquad (65)$$

where H_0^V is the ordinary free Hamiltonian on $\mathscr{F}^V$ constructed out of $-\frac{1}{2}\Delta$ on $L^2(\Lambda^V)$ with periodic boundary conditions (for technical simplicity). It is well known that (65) gives rise to macroscopic occupation of the ground state in any dimension (London 1954). We undertake to prove in this section that this condensation phenomenon persists when an interaction is added, provided that the latter is not too strong (in a sense that we make precise now).

Let $U:\mathbb{R}^3\to\mathbb{R}$ be a positive-definite integrable function, and put

$$H_g^V(\mu)=H_{0,\Delta}^V+\mathscr{U}_g-\mu N^V \qquad (66)$$

where

$$\mathscr{U}_g^{(n)}(x_1,\ldots,x_n)=\sum_{1\leqslant i<j\leqslant n}g^3U(g(x_i-x_j)) \qquad (g>0). \qquad (67)$$

If one takes the infinite volume limit of the system described by the Hamiltonian (66) and afterwards the limit $g \to 0$, one obtains the so-called van der Waals limit in which the system is equivalent to the infinite imperfect Bose gas with interaction parameter $a = \int_{\mathbb{R}^3} \mathrm{d}x\, U(x)$ (de Smedt 1983). Consequently, a model with small g can be considered as a perturbation around the mean-field model. The grand canonical imperfect Bose gas is a sounder basis to perturb around than the grand canonical free Bose gas, because of the pathologies of the latter (see Buffet and Pulé 1983, Ziff *et al* 1977). An alternative possibility would be to perturb around the *canonical free* Bose gas, but we would then lose, together with the use of second quantisation formalism, the possibility of treating the problem in the Bogoliubov approximation.

The mean-field Hamiltonian with a gap is

$$H^V(\mu) = H^V_{0,\Delta} + (a/2V)(N^V)^2 - \mu N^V. \tag{68}$$

The main result of this section is the following theorem.

Theorem 4. For the interacting Bose gas described above (66) one has for μ large enough and g small enough:

$$\lim_{V\to\infty} \inf \left\langle \frac{N_0^V}{V} \right\rangle_\mu^{\text{gc inter}} > 0.$$

Proof. As our main concern in this paper is the Bogoliubov approximation, we shall only sketch the proof of this theorem. If we define

$$\pi_g^V(\sigma) = (\beta V)^{-1} \log \operatorname{Tr}_{\mathscr{F}^V} \exp[-\beta(H_g^V(\mu) + \sigma N_0^V)] \tag{69}$$

$$\pi_g^V(\sigma, x) = (\beta V)^{-1} \log \operatorname{Tr}_{\mathscr{F}^V} \exp(-\beta\{H^V(\mu) + \sigma N_0^V + x[\mathscr{U}_g - a(N^V)^2/2V]\}) \tag{70}$$

then

$$\pi_g^V(\sigma, 0) = (\beta V)^{-1} \log \operatorname{Tr}_{\mathscr{F}^V} \exp[-\beta(H^V(\mu) + \sigma N_0^V)] \equiv \pi_0^V(\sigma) \tag{71}$$

and

$$\pi_g^V(\sigma, 1) = (\beta V)^{-1} \log \operatorname{Tr}_{\mathscr{F}^V} \exp[-\beta(H_g^V(\mu) + \sigma N_0^V)] = \pi_g^V(\sigma). \tag{72}$$

Using (27), one can get the explicit formula

$$\lim_{V\to\infty} \pi_0^V(\sigma) = \begin{cases} p(\mu) & \sigma \geqslant \mu + \Delta - a\rho(\mu) \\ (\mu + \Delta - \sigma)^2/2a + p_0(\sigma - \Delta) & \sigma < \mu + \Delta - a\rho(\mu) \end{cases} \tag{73}$$

where, as before, $p_0(\mu)$ denote respectively the pressure and the density of the infinite free Bose gas without a gap, and $p(\mu)$, $\rho(\mu)$ the pressure and the density of the infinite imperfect Bose gas without a gap (see (11), (12)).

Now, if we put

$$\mu_0 = a\rho_0(-\Delta) - \Delta \tag{74}$$

we have, for $\mu > \mu_0$:

$$\lim_{V\to\infty} -(\mathrm{d}/\mathrm{d}\sigma)\pi_0^V(\sigma)|_{\sigma=0} = \lim_{V\to\infty} \langle N_0^V/V \rangle^{\text{gc,imp}} = (\mu + \Delta)/a - \rho_0(-\Delta) > 0. \tag{75}$$

Using methods similar to those we shall be using in theorem 5 (see also de Smedt 1983), one can prove that for all σ in any arbitrary interval and all $\varepsilon > 0$, there exists a constant C depending on μ only, such that for V large enough and g small enough:

$$|\pi_g^V(\sigma) - \pi_0^V(\sigma)| < g^3 C + \varepsilon. \tag{76}$$

Then, using the convexity of $\pi_g^V(\sigma)$ as a function of σ, we obtain:

$$\left\langle \frac{N_0^V}{V} \right\rangle_\mu^{\text{gc,inter}} = -\frac{d\pi_g^V(\sigma)}{d\sigma}\bigg|_{\sigma=0} \geqslant -\frac{d\pi_0^V(\sigma)}{d\sigma}\bigg|_{\sigma=\Delta/2} - \frac{4(g^3 C + \varepsilon)}{\Delta}. \tag{77}$$

Taking the limit $\liminf_{V\to\infty}$, we obtain the desired result.

Now, denote by $H_{g,\mathrm{B}}^V(\mu, \alpha)$ and $H_\mathrm{B}^V(\mu, \alpha)$ the Bogoliubov appr6ximations to $H_g^V(\mu)$ and $H^V(\mu)$. Introduce the corresponding pressures:

$$P_{g,\mathrm{B}}^V(\mu, \alpha) = (\beta V)^{-1} \log \mathrm{Tr}_{\mathscr{F}_1^V} \exp[-\beta H_{g,\mathrm{B}}^V(\mu, \alpha)] \tag{78}$$

$$P_\mathrm{B}^V(\mu, \alpha) = (\beta V)^{-1} \log \mathrm{Tr}_{\mathscr{F}_1^V} \exp[-\beta H_\mathrm{B}^V(\mu, \alpha)]. \tag{79}$$

It is easy to check that

$$P_\mathrm{B}^V(\mu, \alpha) = -(E_1^V - \Delta - \mu)\alpha^2 - a\alpha^2/2 - a\alpha^4/2V + \tilde{p}_V(\mu - a\alpha^2) \tag{80}$$

where $\tilde{p}_V$ is as in § 2. Therefore

$$P_\mathrm{B}(\mu, \alpha) = \lim_{V\to\infty} P_\mathrm{B}^V(\mu, \alpha) = \Delta\alpha^2 + (\mu\alpha^2 - a\alpha^4/2) + p(\mu - a\alpha^2) \tag{81}$$

with p as in (11). From this the following properties follow: (with μ_0 as in (74)):

(i) when $\mu \leqslant \mu_0$, $\alpha \to P_\mathrm{B}(\mu, \alpha)$ is strictly decreasing;

(ii) when $\mu > \mu_0$, $\alpha \to P_\mathrm{B}(\mu, \alpha)$ takes its maximum at $\alpha_0 = [(\mu - \mu_0)/a]^{1/2}$; it is strictly increasing on $[0, \alpha_0)$ and strictly decreasing on (α_0, ∞).

We see that the presence of a gap suppressed the flat part in the Bogoliubov pressure of the imperfect Bose gas, thus making unambiguous the solution of the condensate equation. As a consequence, the condensate density should be asymptotically degenerately distributed at $(\mu - \mu_0)/a$, independently of the way the infinite volume limit is taken; this is indeed true, and the proof is very much like the one of theorem 1.

Proposition 3. Let $\mathbb{P}_0^V$ denote the probability distribution of N_0^V/V for the imperfect Bose gas with a gap; then one has for any $\varepsilon > 0$

$$\lim_{V\to\infty} \mathbb{P}_0^V\{|N_0^V/V - C| > \varepsilon\} = 0$$

where $C = \max\{0, (\mu - \mu_0)/a\}$, with μ_0 as in (74).

We cannot prove such a strong result for the weakly interacting system (66). We only prove the weaker result that, for μ large enough, $\alpha = 0$ is not a solution of the condensate equation (2), not even in the thermodynamic limit.

Theorem 5. Suppose $\mu > \mu_0$. The following results hold:

(i) if g is sufficiently small, there exists a $\delta > 0$ such that for all V large enough

$$P_{g,\mathrm{B}}^V(\mu, \alpha) < \sup_{\alpha' \in \mathbb{R}} P_{g,B}^V(\mu, \alpha') \qquad \forall \alpha \in [0, \delta];$$

(ii) if the limit $\lim_{V\to\infty} P^V_{g,\mathrm{B}}(\mu, \alpha)$ exists for all α and is equal to $P_{g,\mathrm{B}}(\mu, \alpha)$ and if g is small enough, there exists a $\delta > 0$ such that

$$P_{g,\mathrm{B}}(\mu, \alpha) < \sup_{\alpha' \in \mathbb{R}} P_{g,\mathrm{B}}(\mu, \alpha') \qquad \forall \alpha \in [0, \delta].$$

Proof. The proof consists in finding upper and lower bounds for $P^V_{g,\mathrm{B}}(\mu, \alpha)$ and comparing the upper bound of $P^V_{g,\mathrm{B}}(\mu, \alpha)$ on the interval $[0, \delta]$ with the supremum over the lower bound of $P^V_{g,\mathrm{B}}(\mu, \alpha)$.

To obtain the upper bound we use the superstability condition in de Smedt *et al* (1983), de Smedt (1983) namely for all $\varepsilon_1 > 0$ there exists a V_0 such that for all $V > V_0$:

$$\mathcal{U}_g \geq a(N^V)^2/2V(1+\varepsilon_1) - g^3 bN^V \tag{82}$$

where $b = U(0)$.

From this, it follows easily that we have for all $\varepsilon > 0$ and for all α on an arbitrary compact interval

$$P^V_{g,\mathrm{B}}(\mu, \alpha) \leq P_\mathrm{B}(\mu + g^3 b, \alpha) + \varepsilon \tag{83}$$

if V is large enough.

Now, we look for a lower bound for $P^V_{g,\mathrm{B}}(\mu, \alpha)$. As before, denote by $A_\mathrm{B}(\alpha)$ the Bogoliubov approximation of the operator A and define

$$\hat{p}^V(x) = (\beta V)^{-1} \log \mathrm{Tr}_{\mathcal{F}_1^V} \exp\Big(-\beta\{H^V_\mathrm{B}(\mu + g^3 b, \alpha) + x[\mathcal{U}_{g\mathrm{B}}(\alpha) - a(N^V_\mathrm{B}(\alpha))^2/2V + g^3 bN^V_\mathrm{B}(\alpha)]\}\Big). \tag{84}$$

Then

$$\hat{p}^V(0) = P^V_\mathrm{B}(\mu + g^3 b, \alpha) \tag{85}$$

$$\hat{p}^V(1) = P^V_{g,\mathrm{B}}(\mu, \alpha). \tag{86}$$

Moreover $\hat{p}^V(x)$ is a convex function of x. Thus

$$\hat{p}^V(1) - \hat{p}^V(0) \geq \hat{p}^{V\prime}_+(0) \tag{87}$$

or

$$P^V_{g,\mathrm{B}}(\mu, \alpha) \geq P^V_\mathrm{B}(\mu + g^3 b, \alpha) + \hat{P}^{V\prime}_+(0). \tag{88}$$

Now, if

$$\tilde{\omega}^V_\mu(A) = \frac{\mathrm{Tr}_{\mathcal{F}_1^V} A \exp\{-\beta[\tilde{H}^V_0 - \mu\tilde{N}^V + a(\tilde{N}^V)^2/2V]\}}{\mathrm{Tr}_{\mathcal{F}_1^V} \exp\{-\beta[\tilde{H}^V_0 - \mu\tilde{N}^V + a(\tilde{N}^V)^2/2V]\}} \tag{89}$$

then

$$\hat{p}^{V\prime}_+(0) = -V^{-1}\tilde{\omega}^V_{\mu+g^3b-a\alpha^2}(\mathcal{U}_{g\mathrm{B}}(\alpha)) + (a/2V^2)\tilde{\omega}^V_{\mu+g^3b-a\alpha^2}[(N^V_\mathrm{B}(\alpha))^2] - (g^3 b/2V)\tilde{\omega}^V_{\mu+g^3b-a\alpha^2}(N^V_\mathrm{B}(\alpha)). \tag{90}$$

Assume furthermore that α is such that $\mu + g^3 b - a\alpha^2 < a\rho_\mathrm{c}$ or $a\alpha^2 > \mu + g^3 b - a\rho_\mathrm{c} \equiv \gamma$. Let μ_1 satisfy $\mu_1 = (\mu + g^3 b - a\alpha^2) - a\rho_0(\mu_1)$ and let

$$F_{3/2}(s, \mu) = \sum_{n=1}^{\infty} \frac{\exp(n\beta\mu)\exp(-s^2/2n\beta)}{(n)^{3/2}}. \tag{91}$$

One can prove as in de Smedt (1983), that

$$\lim_{V\to\infty}\{-[\tilde{\omega}^{V}_{\mu+g^3b-a\alpha^2}(\mathcal{U}_{gB}(\alpha)]/V\} = -\tfrac{1}{2}a\alpha^4 - a\alpha^2\rho_0(\mu_1) - \tfrac{1}{2}a(\rho_0(\mu_1))^2$$
$$-\frac{g^3}{(2\pi\beta)^{3/2}}\alpha^2\int_{\mathbb{R}^3}\mathrm{d}x\,U(gx)F_{3/2}(\|x\|,\mu_1)$$
$$-\frac{g^3}{2(2\pi\beta)^3}\int_{\mathbb{R}^3}\mathrm{d}x\,U(gx)[F_{3/2}(\|x\|,\mu_1)]^2 \qquad (92)$$

($\|x\|$ denotes the norm of the vector $x\in\mathbb{R}^3$)

$$\lim_{V\to\infty}\tfrac{1}{2}a\tilde{\omega}^{V}_{\mu+g^3b-a\alpha^2}[(N^V_{\mathrm{B}}(\alpha))^2/V^2] = \tfrac{1}{2}a\alpha^4 + a\alpha^2\rho_0(\mu_1) + \tfrac{1}{2}a(\rho_0(\mu_1))^2 \qquad (93)$$

$$\lim_{V\to\infty} -g^3b\tilde{\omega}^{V}_{\mu+g^3b-a\alpha^2}(N^V_{\mathrm{B}}(\alpha)/V) = -g^3b\alpha - g^3b\rho_0(\mu_1). \qquad (94)$$

Thus, for all $\alpha > (\gamma/a)^{1/2}$ and all $\varepsilon > 0$, there exists a V_0 such that for $V > V_0$, we have

$$P^V_{g,\mathrm{B}}(\mu,\alpha) \geqslant P_{\mathrm{B}}(\mu+g^3b,\alpha) - f(g,\alpha) - \varepsilon \qquad (95)$$

where

$$f(g,\alpha) = -g^3b\rho_0(\mu_1) - g^3b\alpha^2 - \frac{g^3\alpha^2}{(2\pi\beta)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}x\,U(gx)F_{3/2}(\|x\|,\mu_1)$$
$$-\frac{g^3}{2(2\pi\beta)^3}\int_{\mathbb{R}^3}\mathrm{d}x\,U(gx)(F_{3/2}(\|x\|,\mu_1))^2. \qquad (96)$$

Note that $f(g,\alpha)\to 0$ as $g\to 0$. Since $\alpha\to P_{\mathrm{B}}(\mu+g^3b,\alpha)$ is strictly increasing up to $\alpha_1 = [(\mu+g^3b-\mu_0)/a]^{1/2}$, then, if $\varepsilon>0$ is sufficiently small, there is a δ satisfying $0<\delta<\alpha_1$ such that

$$P_{\mathrm{B}}(\mu+g^3b,\alpha) < P_{\mathrm{B}}(\mu+g^3b,\alpha_1) - 4\varepsilon \qquad (97)$$

provided $\alpha<\delta$. Also, if we choose g such that $f(g,\alpha_1)<\varepsilon$, as $\alpha_1 > (\gamma/a)^{1/2}$, we obtain using (95)

$$P^V_{g,\mathrm{B}}(\mu,\alpha_1) \geqslant P_{\mathrm{B}}(\mu+g^3b,\alpha_1) - 2\varepsilon \qquad (98)$$

for V large enough. Combining (83), (97) and (98) we have for sufficiently large V

$$P^V_{g,\mathrm{B}}(\mu,\alpha) < P^V_g, \mathrm{B}(\mu,\alpha_1) - \varepsilon$$

if $\alpha<\delta$, which implies both statements in the theorem.

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Appendix

We state without proof the following version of Laplace's theorem due to Lewis and Pulé (1983b).

Generalised Laplace theorem. Suppose that

(i) $\forall V > 0 : g^V : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions with $g(0) = A > -\infty$;

(ii) $g^V \to g$ uniformly on compacts;

(iii) $d\mu^V(x)$ is a measure, being either the Lebesgue measure or $dm(\beta V x)$ where $m(x) = \max\{n \in \mathbb{N}: n \leqslant x\}(\beta > 0)$.

Then, if $\mu_\infty = \lim_{V\to\infty}(\liminf_{x\to\infty} x^{-1}(\inf_{V'>V} g^{V'}(x)))$, we have for all $\mu < \mu_\infty$

$$\lim_{V\to\infty} \frac{1}{\beta V} \log \int_0^\infty \exp[\beta V(\mu x - g^V(x))]\, d\mu^V(V) = \sup_{x\in\mathbb{R}^+} (\mu x - g(x)).$$

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